HIGHER QUANTUM CONSERVED CURRENT IN A NEW COMPLETELY INTEGRABLE MODEL

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ABSTRACT. The first higher local quantum conserved current in the recently proposed new completely integrable $(2\,e^{\beta\varphi} + e^{-2\beta\varphi})_2$ model is explicitly constructed thus proving absence of particle production and factorization of multiparticle scattering.

1. The existence of infinite series of higher conservation laws in two-dimensional completely integrable field theory models plays a fundamental role for their exact solvability on quantum level [1, 2]. They impose quite non-trivial restrictions on the dynamics:

$$\sum_{\text{in}} (p_{\xi,\eta})^{2k+1} = \sum_{\text{out}} (p'_{\xi,\eta})^{2k+1}, \quad k = 0, 1, 2, \ldots \tag{1}$$

where the sums run over the sets of in- and out-particle momenta for an arbitrary scattering process. The subscripts ξ, η indicate light-cone components $(\xi, \eta = \frac{1}{2}(x^0 \pm x^1),\ A_{\xi,\eta} = A_0 \pm A_1,$ $A_\mu B^\mu = \frac{1}{2}(A_\xi B_\eta + A_\eta B_\xi)$ etc.). Both conservation laws (for each k) are connected through space reflection. The crucial consequence of (1) are:

(i) Absence of multiparticle production
(ii) Factorization [3, 4] of each scattering process into a sequence of successive two-particle collisions.

In particular, from the first non-trivial higher local quantum conserved current (HLQCC) giving (1) with $k = 1$ the so-called factorization equations for two-particle amplitudes which follow, lead to the exact determination of the total quantum S-matrix [1].

Recently it was shown [5] that the following model with Lagrangian:

$$L(x) = \tfrac{1}{2}(\partial_\mu\varphi)^2 - m^2/6\beta^2(2\,e^{\beta\varphi} + e^{-2\beta\varphi}) \tag{2}$$

($\varphi(x)$ being a two-dimensional scalar field) is completely integrable (on classical level) and

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possesses an infinite number of local conserved currents. Subsequently the exact quantum S-matrix of (2) was found [6] assuming validness of features (i), (ii). In the present note our aim is to construct explicitly the first HLQCC of (2) from which (i), (ii) are deduced, thus providing complete dynamical justification of the proposed exact S-matrix [6].

2. We shall follow the method developed in [7, 8] which is based on the normal product formalism in the Bogoliubov–Parasiuk–Hepp–Zimmermann renormalization scheme [9]. The main tools are proper renormalized quantum equations of motion (QEM) [10] and Zimmermann identities (ZIs) [9] for renormalized composite field operators. Furthermore, we shall put our discussion on a more general ground and consider the whole class of two-dimensional scalar field models:

$$L(x) = \tfrac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi), \qquad V(\varphi) \equiv \tfrac{1}{2}m^2\varphi - L_I(\varphi) \tag{3}$$

($V(\varphi)$ arbitrary entire function in φ) in the context of searching for which of them possess HLQCCs yielding (1) (and consequently, appearing to be exactly solvable). In the case of $k = 1$ in (1) the answer is known [3]: the only one such model is the Sine–Gordon ($V(\varphi) = m^2/\beta^2$ $(1 - \cos\beta\varphi)$) model. The same statement is true for the supersymmetric generalization of the latter [8]. That is why we concentrate on the next non-trivial HLQCC giving rise to $k = 2$ in (1). As in [7, 8] arguments of Lorentz covariance and field dimensions lead to the following general structure of the sought HLQCC in light-cone components:

$$\begin{aligned} J_\xi^{(5)} &= N[\varphi_{\xi\xi\xi}^2 + \alpha_1\varphi_{\xi\xi}^2 + \alpha_2\varphi_\xi^2\varphi_{\xi\xi}^2 + \alpha_3\varphi_\xi^6]\,(x), \\ J_\eta^{(5)} &= N[A_1(\varphi)\varphi_{\xi\xi\xi\xi} + A_2(\varphi)\varphi_\xi\varphi_{\xi\xi\xi} + A_3(\varphi)\varphi_{\xi\xi}^2 + \\ &\quad + A_4(\varphi)\varphi_\xi^2\varphi_{\xi\xi} + A_5(\varphi)\varphi_\xi^4](x) \end{aligned} \tag{4}$$

($\varphi_\xi \equiv \partial_\xi\varphi$, etc.) and analoguously for the space-reflected current by means of the change $\xi \to \eta$. Here and below the symbol N denotes canonical (minimally subtracted) normal product [9]. The constants α_i, $i = 1, 2, 3$ and the (entire) functions $A_j(\varphi)$, $j =$ i, ..., 5 *together with the form of* $V(\varphi)$ (3) are to be determined by the requirement that the corresponding Ward identity for the conservation of $J^{(5)}$ holds ($\langle\cdots\rangle$ denotes connected time-ordered Green functions):

$$\begin{aligned} &\partial_\eta\langle J_\xi^{(5)}(x)X\rangle + \partial_\xi\langle J_\eta^{(5)}(x)X\rangle = \\ &\qquad -i\sum_{l=1}^{L}\{\partial_\xi^2\,\delta^{(2)}(x - x_l)\langle\varphi_{\xi\xi\xi}(x)\hat{X}^l\rangle + \\ &\qquad + \partial_\xi\delta^{(2)}(x - x_l)\langle N[3\alpha_1\varphi_{\xi\xi}^2 + 2\alpha_2\varphi_\xi^2\varphi_{\xi\xi}](x)\hat{X}^l\rangle + \\ &\qquad + \delta^{(2)}(x - x_l)\langle N[2\alpha_2\varphi_\xi\varphi_{\xi\xi}^2 + 6\alpha_3\varphi_\xi^5](x)\hat{X}^l\rangle\}; \end{aligned} \tag{5}$$

where

$$X \equiv \prod_{l=1}^{L} \varphi(x_l), \qquad \hat{X}^l \equiv \prod_{r=1,\, r\neq l}^{L} \varphi(x_r).$$

To this end we insert eqns. (4) into eqn. (5) and make use of the QEM (cf. [7]). ($Q(\partial_x)$ is an arbitrary differential monomial; P is an arbitrary composite field):

$$\langle N[PQ(\partial_x)(\varphi_{\xi\eta} + \partial V/\partial\varphi)](x)X\rangle = -i \sum_{l=1}^{L} [Q(\partial_x)\delta^{(2)}(x - x_l)] \times$$

$$\times \langle N[P](x)\hat{X}^l\rangle + m^2 \langle N[PQ(\partial_x)(\varphi - \{\varphi\})](x)X\rangle. \qquad (6)$$

The curly brackets inside the N-symbol are the last 'anomalous' anisotropic normal product term on the r.h.s. of (6) and indicate two oversubtractions in each subdiagram containing the corresponding φ-line. It is well known that it is just this 'anomalous' term in QEM which is responsible for (in some cases as in the $O(N)$ non-linear sigma-model [11] crucial) the differences between the quantum conserved currents and their classical counterparts. The anisotropic normal products are to be expanded in terms of canonical ones by means of the ZIs. It was shown in [7] that because of the superrenormalizability of models (3) and due to the peculiarity of two-dimensional Lorentz kinematics, the only graphs contributing to the ZIs are those with only one interaction (L_I-) vertex. In our case the number of the relevant graphs are 17. Some of them giving rise to the ZI for the 'anomalous' term:

$$3m^2\alpha_1 N[\varphi^2_{\xi\xi}\partial_\xi(\varphi - \{\varphi\})](x) = -3\alpha_1 c_1 N[\partial^5_\xi(V^{(3)})](x) -$$

$$- 6\alpha_1 c_2 N[\varphi_{\xi\xi}\partial^3_\xi(V^{(2)})](x); \quad V^{(n)} \equiv \partial^n V/\partial\varphi^n, \qquad (7)$$

are displayed in Fig. 1 ($D^\omega = t_p^\omega - t_p^{\omega-1}$ acts on the corresponding box – subgraph t_p^ω being the conventional Taylor operator [9]; bars on the line denote ∂_ξ-derivatives).

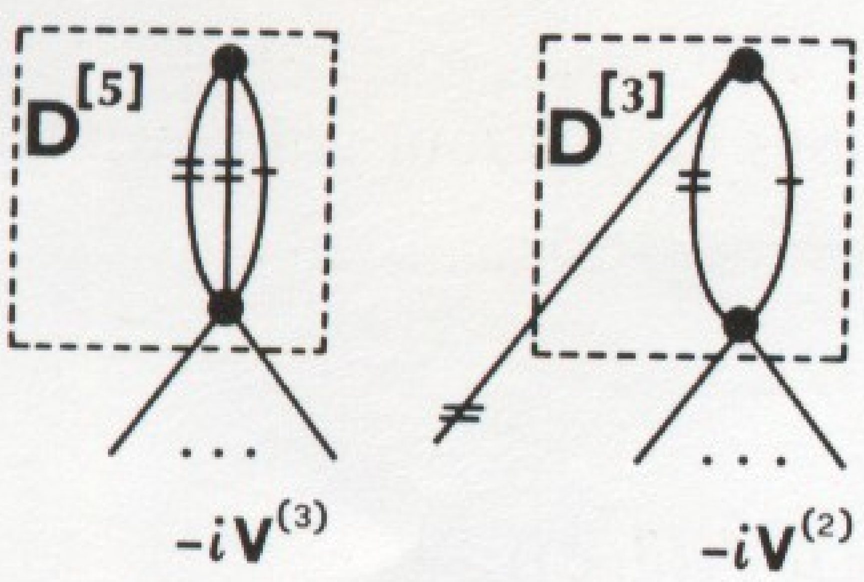

Fig. 1. Graphs contributing to the ZI (7).

$c_{1,2}$ (and all other c_r, $r = 0, 3, 4, ..., 16$, arising in the remaining ZIs) are computable constants. Accounting for the above facts we reduce eqn. (5) to the following one:

$$\sum_{n=1}^{7} \langle N[F_n(A, V; \alpha, c)G_n(\varphi_\xi)](x)X\rangle = 0;$$

$$G_n(\varphi_\xi) = \{\varphi_{\xi\xi\xi\xi\xi\xi}, \varphi_\xi\varphi_{\xi\xi\xi\xi\xi}, \varphi_{\xi\xi}\varphi_{\xi\xi\xi\xi}, \varphi^2_\xi\varphi_{\xi\xi\xi\xi}, \varphi_\xi\varphi^2_{\xi\xi\xi}, \varphi^3_\xi\varphi_{\xi\xi\xi}, \varphi^5_\xi\} \qquad (8)$$

where the functions F_n are linear in $A_j(\varphi), A_j'(\varphi)$ and $V^{(k)}(\varphi)$ $(k \geqslant 1)$ with coefficients linearly dependent on c_r, α_i. As a consequence of (8) we have seven linear ordinary differential equations:

$$F_n(A, V; \alpha, c) = 0. \tag{9}$$

Solving (9) for the A_j's we are left with two ordinary differential equations for $V'(\varphi)$:

$$\begin{aligned}
&6\alpha_3 V' + [-\tfrac{1}{2}\alpha_2 + 30\alpha_3 c_{16}]V^{(3)} + [\tfrac{1}{2} - \alpha_2(c_6 + c_{11} - c_7) + 45\alpha_3 c_{15}]V^{(5)} - \\
&\quad - \tfrac{3}{2}\alpha_1 c_2 V^{(6)} + [-\tfrac{1}{4}c_0 - \tfrac{1}{2}\alpha_2(c_4 + 2c_9 - 2c_5 - c_{10}) + 30\alpha_3 c_{14}]V^{(7)} - \\
&\quad - \tfrac{3}{4}\alpha_1 c_1 V^{(8)} + [-\tfrac{1}{2}\alpha_2(c_3 + c_8) + \tfrac{15}{2}\alpha_3 c_{13}]V^{(9)} - \tfrac{3}{2}\alpha_3 c_{12} V^{(11)} = 0,
\end{aligned} \tag{10a}$$

$$\begin{aligned}
&2\alpha_2 V' + 3\alpha_1 V^{(2)} + [-5 + 4\alpha_2(c_6 + c_{11} - c_7)]V^{(3)} + 15\alpha_1 c_2 V^{(4)} + \\
&\quad + [\tfrac{5}{2}c_0 - 120\alpha_3 c_{14} + 5\alpha_2(c_4 + 2c_9 - 2c_5 - c_{10})]V^{(5)} + \tfrac{15}{2}\alpha_1 c_1 V^{(6)} + \\
&\quad + [5\alpha_2(c_3 + c_8) - 75\alpha_3 c_{13}]V^{(7)} + 15\alpha_3 c_{12} V^{(9)} = 0.
\end{aligned} \tag{10b}$$

Now eqns. (10) tell us that:

$$V(\varphi) = \sum_a B_a \, e^{\lambda_a \varphi}, \qquad B_a = \text{const.} \tag{11}$$

Furthermore, it is easy to see that we can have at most *two* different $(\lambda_1 \neq \lambda_2)$ coupling constants in (11) because of the *overdetermined* system (10). Namely, taking $\lambda_1 \equiv \beta$ as a free parameter, we must substitute*

$$V(\varphi) = \tilde{m}^2(\beta - \lambda_2)^{-1}\left(\frac{1}{\beta}\, e^{\beta\varphi} - \frac{1}{\lambda^2}\, e^{\lambda_2 \varphi}\right) \tag{12}$$

into eqns. (10a, b) thus obtaining four algebraic equations for the four unknown quantities α_i, $i = 1, 2, 3$, and λ_2 as functions of β.

We find two types of solutions:

$$\begin{aligned}
&\tilde{\alpha}_1 = 0, \qquad \tilde{\lambda}_2 = -\beta, \qquad \tilde{\alpha}_2 = \tfrac{5}{2}\beta^2(1 + \beta^2 \tilde{f}_2(\beta, c)), \\
&\tilde{\alpha}_3 = \tfrac{1}{8}\beta^4(1 + \beta^2 \tilde{f}_3(\beta, c)),
\end{aligned} \tag{13a}$$

* The coefficients $B_{1,2}$ are uniquely determined by the requirement on absence of a linear term in (11) and that the quadratic term should acquire the conventional mass term form. The quantity $\tilde{m}^2 = m^2 h(m/\mu)$ contains the (finite) mass counter-term additions accounting for the freedom in subtracting the divergent point-loops, μ being an arbitrary subtraction point. The normalization of $h(m/\mu)$ is such that m is precisely the physical mass of the fundamental particles.

i.e. the Sine (sinh)–Gordon model, and:

$$\alpha_1 = -\tfrac{5}{3}\beta(1+\beta^2 f_1(\beta, c)), \qquad \alpha_2 = 5\beta^2(1+\beta^2 f_2(\beta, c)),$$
$$\alpha_3 = \tfrac{1}{3}\beta^4(1+\beta^2 f_3(\beta, c)), \qquad \lambda_2 = -2\beta(1+\beta^2 g(\beta, c)), \tag{13b}$$

where all functions $\tilde{f}_i(\beta, c)$, $f_i(\beta, c)$, $g(\beta, c)$ vanish if we set $c_r = 0$, $r = 0, 1, \ldots, 16$, i.e. in the classical theory (we omit their explicit forms for brevity).

Thus, on classical level $(c_r \to 0,\ \tilde{m}^2 \to m^2)$ solution (12), (13b) is just the model (2). Moreover, because of its superrenormalizability there is no coupling constant renormalization and, consequently, $g(\beta, c)$ (which is quantum correction to λ_2 due to the ZIs (7) etc.) must vanish, i.e. $\lambda_2 = -2\beta$ in (13b) and we recover (2) also on quantum level. The only new point is that the coefficients α_i, $i = 1, 2, 3$ (13b) in the quantum current (4) (as well as $A_j(\varphi)$, $j = 1, \ldots, 5$) acquire quantum corrections as compared to their classical counterparts.

3. The remaining HLQCCs ($k \geqslant 3$) of (2) cannot be constructed according to the above scheme, since (2) has only a finite number of conserved currents polynomial in $\varphi_\xi, \varphi_{\xi\xi}$ etc. [12], unlike the Sine–Gordon model. However, it turns out that the HLQCC (4) alone is sufficient to prove absence of multiparticle production and factorization of scattering of the fundamental particles in the model.

Consider a general two-particle collision process $2 \to N$ in the center-of-mass frame of the incoming particles. Equations (1) for $k = 0$ (energy-momentum conservation) and $k = 2$ (conservation of (4)) give:

$$\frac{q+1}{q} = \sum_{l=1}^{N} q'_l = \sum_{l=1}^{N} \frac{1}{q'_l}, \tag{14a}$$

$$\frac{q^5+1}{q^5} = \sum_{l=1}^{N} q'^5_l = \sum_{l=1}^{N} \frac{1}{q'^5_l}; \tag{14b}$$

$$q \equiv p_{1\xi}/m = m/p_{1\eta} = p_{2\eta}/m = m/p_{2\xi} > 0, \qquad q'_l \equiv p_{l\xi}/m = m/p_{l\eta} > 0$$

(recall that on mass shell $p^2 = p_\xi p_\eta = m^2$). Clearly, eqns. (14) exclude the cases $N = 1, 3$. Now using the identities:

$$x^5 + 1/x^5 = (x+1/x)^5 - 5(x+1/x)^3 + 5(x+1/x),$$

$$\left(\sum_{i=1}^{N} x_i\right)^5 = \sum_{i=1}^{N} x_i^5 + n\left(\sum_{i=1}^{N} x_i\right)^3 \sum_{k<l}^{N} x_k x_l + P_n(x_i)$$

where $n = 2$ if $N \geqslant 5$, $n = 20/9$ if $N \leqslant 4$ and $P_n(x_i)$ is a fifth degree polynomial in x_i strictly positive for $x_i > 0$ we obtain from (14) the equality:

$$\left(\sum_{l=1}^{N} q'_l\right)^3 \left(5 - n \sum_{k<l}^{N} (q'_k q'_l)^{\pm 1}\right) = 5 \sum_{l=1}^{N} q'_l + P_n(q'_l)$$

or, equivalently:

$$\frac{5}{n} > \sum_{k<l}^{N} q'_k q'_l = \sum_{k<l}^{N} (q'_k q'_l)^{-1}. \tag{15}$$

If we relabel the out-particles in such a way that $q'_1 \leqslant q'_2 \leqslant \ldots \leqslant q'_N$ we can deduce from (15) the inequality:

$$5/n > q'_{N-1} q_N > n/5\, N(N-1)/2, \quad \text{i.e.}\ N(N-1) < 2(5/n)^2. \tag{16}$$

Thus with the above choice of n the only possible solution of (16) is $N = 2$.

Finally, having at hand the conserved higher charge, generated by (4) and the already proved absence of multiparticle production one can construct the argument to establish factorization in strict analogy with the Sine–Gordon (massive Thirring) case [13] where only the lowest non-trivial HLQCC [7, 14] (corresponding to $k = 1$ in eqns. (1)) was needed.

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NOTE ADDED IN PROOF

The statement of ref. [12] about the finiteness of the number of higher local conserved currents of (3), polynomial in the field and its derivatives, quoted in the text is not correct. For the general construction of the classical currents in the framework of the inverse scattering method, see A.V. Mikhailov in the 'Proceedings of the American–Soviet Soliton Workshop', Kiev 1979, to be published by North Holland. The author thanks A.V. Mikhailov and P.P. Kulish for drawing his attention to the above fact.